A SHORT PROOF OF THE LEVY CONTINUITY THEOREM IN HILBERT SPACE*

BY

J. FELDMAN

ABSTRACT

A short proof of the Levy continuity theorem in Hilbert space.

In the theory of the normal distribution on a real Hilbert space H, certain functions ϕ have been shown by L. Gross to give rise to random variables $\phi\sim$ in a natural way; in particular, this is the case for functions which are "uniformly τ-continuous near zero". Among such functions are the characteristic functions ϕ of probability distributions m on H, given by $\phi(y) = \int e^{i\langle y,x\rangle}dm(x)$. The following analogue of the Levy continuity theorem has been proved by Gross: Let ϕ_j be the characteristic function of the probability measure m_j on H, Then necessary and sufficient that $\int f\,dm_j \to \int f\,dm$ for some probability measure m and all bounded continuous f, is that there exists a function ϕ, uniformly τ-continuous near zero, with $\phi_j\sim \to \phi\sim$ in probability. ϕ turns out, of course, to be the characteristic function of m. In the present paper we give a short proof of this theorem.

Let H and K be a pair of real linear spaces in duality, and let $\mathscr{S}$ be the smallest σ-field of subsets of H for which all elements of K become measurable. If m is a probability measure on $\mathscr{S}$, then the formula

$$\phi(y) = \int e^{i\langle x,y\rangle}dm(y)$$

defines a function on K, the "characteristic function" of m. ϕ is clearly positive-definite, 1 at the origin, and weakly continuous. One is then led to consider the possibility of generalizing theorems about characteristic functions which are known for the finite-dimensional case. Two basic theorems are the following.

THEOREM I. (S. BOCHNER) *The characteristic functions of probability measures on finite-dimensional spaces are precisely the continuous positive-definite functions which are* 1 *at the origin.*

THEOREM II. (P. LEVY) *Let* ϕ_j, $j = 1,2,\cdots$ *be characteristic functions of probability measures* m_j *on the finite-dimensional space* K, *and let* ϕ *be a continuous function on* K *with* $\phi(0) = 1$. *A necessary and sufficient condition that* ϕ *be the characteristic function of a measure* m *such that* $\int f dm_j \to \int f dm$ *for all continuous bounded* f, *is that* $\phi_j \to \phi$ *Lebesgue-almost-everywhere.*

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The first of these theorems has been generalized to Hilbert space H, in the following form. A weaker topology $\mathscr{T}$ is placed on H (see [1,2,6]) via the semi-norms $\{\|\cdot\|_A : A$ any Hilbert-Schmidt operator on $H\}$, where $\|x\|_A = \|Ax\|$. Then we have

THEOREM 1. *Necessary and sufficient conditions for a function ϕ on Hilbert space H to be a characteristic function are that ϕ be positive-definite,* 1 *at* 0, *and uniformly continuous in the topology $\mathscr{T}$ (and in fact the continuity assumption may be weakened to $\mathscr{T}$-continuity at* 0).

This theorem has been proven by V. Sazanov [6], in a somewhat different form by R.A. Minlos [4], and also by L. Gross [2] as a corollary to the next theorem. Recently K. Ito has given an extremely direct and simple proof, as yet unpublished.

Theorem II has also been generalized to Hilbert space by L. Gross [2]. His proof is rather long and technical, as befits a first proof. The object of the present note is to give a simpler and more transparent proof. One of the devices in my proof, the use of the measure

$$e^{-\|x\|^2/2}\,dm(x)$$

to estimate the m-measure of a set, was already used by K. Ito in his brief proof of Theorem 1, and by Kolmogorov, [3]; it goes back to Prokhorov, [5]. In order to state Gross's generalization, we recall some definitions and results from [7] and [2].

By a *cylinder function* on H we shall mean a function f such that $f = f \circ P$ for some finite-dimensional projection P. Let $\mathscr{A}$ be the *-algebra of all continuous cylinder functions on H, and $\mathscr{A}_\infty$ the bounded elements of $\mathscr{A}$. A linear functional on $\mathscr{A}_\infty$ may be defined by integrating f over PH with respect to the normal distribution on PH, i.e., $\int f\,dn$ is defined by

$$\int f\,dn = \frac{1}{\sqrt{(2\pi)^d}} \int_{PH} f(x)e^{-1/2\|x\|^2}dx,$$

where d = dimension of PH. This is to be interpreted as integration with respect to a generalized normal distribution on H. Although many different P may be used, the number $\int f\,dn$ is independent of the P chosen. It is now possible, as in [7], to define a homomorphism, which in this case is clearly an injection, from $\mathscr{A}$ into random variables on some probability space, call this isomorphism $f \to f^\sim$, such that if f is bounded, then

$$\int f\,dn = \int f^\sim d\mathrm{Pr}.$$

The isomorphism preserves the algebraic operations including complex conjugation, and for any bounded continuous complex function u of a complex variable, sends $u(f)$ to $u(f^{\sim})$.

In [1,2] Gross has extended the injection $f \to f^{\sim}$ to a larger class of functions; those which are uniformly *$\mathscr{T}$-continuous near zero*. This term means, for a function f, that there is a sequence A_j of Hilbert-Schmidt operators with $\mathrm{tr}(A_j^* A_j) \to 0$ and such that f is uniformly $\mathscr{T}$-continuous on $\{x: \| A_j x \| \leqq 1\}$. We shall denote by $\mathscr{B}$ the functions satisfying this condition, and by $\mathscr{B}_\infty$ the bounded ones. For any net of finite-dimensional projections $P_k \uparrow I$, it is shown in [1, 2] that $f^{\sim} \equiv \lim$ in probability $(f \circ P_k)^{\sim}$ exists. The extended map again is an injection preserving algebraic operations and conjugation, and one may define $\int f dn = \int f^{\sim} d\mathrm{Pr}$ on bounded elements. Furthermore, for continuous u with compact support, f in $\mathscr{B}$ implies $u(f)$ is in $\mathscr{B}_\infty$.

We are now in a position to state Gross's generalization of Theorem II.

THEOREM 2. *Let ϕ_j, $j = 1,2,\cdots$ be the characteristic function of a probability measure m_j on the Hilbert space H. Then necessary and sufficient for the measures m_j to converge to a probability measure m, in the sense that $\int f dm_j \to \int f dm$ for all bounded continuous f, is that the random variables $\phi_j^{\sim}$ converge in probability to $\phi^{\sim}$, for some ϕ in $\mathscr{B}$ with $\phi(0) = 1$. ϕ then turns out to be the characteristic function of m.*

Proof. All we shall show here is that if ϕ is in $\mathscr{B}$ with $\phi(0) = 1$, and $\phi_j^{\sim} \to \phi^{\sim}$ in probability, then the set $\{m_j : j = 1,2,\cdots\}$ is precompact. The "necessity" proof, and the remainder of the proof of "sufficiency", are fairly straightforward in Gross's original treatment.

First we note that $|\phi| \leqq 1$. For let u be any nonnegative continuous function of a complex variable, having compact support, and vanishing in the unit disk. Then each $u(\phi_j) = 0$, and $\int u(\phi_j)dn \to \int u(\phi)dn$, so $\int u(\phi)dn = 0$. Thus $u(\phi)^{\sim} = 0$, hence $u(\phi) = 0$. Thus $|\phi| \leqq 1$.

Next, observe that for given $a > 0$, $b > 0$, finite-dimensional projection P, and $Q = I - P$, the set $\{x: \|(aP + Q)x\| \leqq b\}$ is contained in the b-neighborhood in H of the b/a-sphere in the range of P. Thus, by a theorem of Prokhorov, all we need show is that for any preassigned $b > 0 \,\exists\, a > 0$ and finite-dimensional P such that, for all sufficiently large j,

$$m_j\{x: \|(aP + Q)x\| \leqq b\} > 1 - b.$$

Select $b > 0$, and let $c = (1 - e^{-b^2/2})b/3$. There exists a Hilbert-Schmidt operator $A \geqq 0$ such that $\|Ax\| < 1$ implies $|1 - \phi(x)| < c$. Then

$$\mathrm{re}\,\phi(y) > 1 - c - 2\|Ay\|^2.$$

Let $T = aP + Q$, where $0 < a < 1$, and P is a finite-dimensional projection. Then

$$\int_H \operatorname{re}\phi \circ T dn > 1 - c - 2\operatorname{tr}(AT^2A).$$

Now, a simple finite-dimensional calculation shows that, for $f \in \mathscr{A}_\infty$, $\int f \circ T dn = \int f g dn$, where

$$g(x) = \frac{1}{(\sqrt{2\pi a})^d} e^{(1-a^{-2}/2)\|Px\|^2},$$

d being the dimension of P. Then the same equation holds for $f \in \mathscr{B}_\infty$. (This really amounts to saying $(dn \circ T^{-1}/dn) = g$; see [7] for a fuller discussion of these ideas). Now: since $\phi_j^\sim \to \phi^\sim$ in probability, it follows that $\phi_j^\sim g^\sim \to \phi^\sim g^\sim$ in probability, so $\int \phi_j^\sim g^\sim d\operatorname{Pr} \to \int \phi^\sim g^\sim d\operatorname{Pr}$. Then, for sufficiently large j,

$$\operatorname{re}\int \phi_j \cdot T dn = \operatorname{re}\int \phi_j g dn > 1 - c - 2\operatorname{tr}(AT^2A).$$

Now let Q_k be a net of finite-dimensional projections ascending to Q, and $P_k = P + Q_k$. Then, for fixed j,

$$\int \phi_j \circ P_k T dn = \int \phi_j \circ T P_k dn \to \int \phi_j \circ T dn.$$

Now, $\phi_j \circ P_k T$ restricted to $P_k H$ is the characteristic function of the measure $m_j \circ (P_k T)^{-1}$ on $P_k H$, so

$$\int \phi_j \circ P_k T dn = \int_{P_k H} e^{-(\|x\|^2/2)} dm_j \circ (P_k T)^{-1}$$

$$= \int_H e^{-(1/2)\|P_k T x\|^2} dm_j(x).$$

Then, taking limits in k, $\int \phi_j \circ T dn = \int e^{-(1/2)\|Tx\|^2} dm_j(x)$.

$$\text{Now: } \int e^{-(1/2)\|Tx\|^2} dm_j(x) < m_j\{x : \|Tx\| \leqq b\} + e^{-(b^2/2)} m_j\{x : \|Tx\| > b\}.$$

Thus, $m_j\{x : \|Tx\| \leqq b\} > 1 - b/3 - (1/1 - e^{-(b^2/2)})\, 2\operatorname{tr}(AT^2A)$. Since $\operatorname{tr}(AQA) \downarrow 0$ as $P \uparrow I$, a finite-dimensional P may be chosen with $2\operatorname{tr}(AQA) < c$. Then choose a so small that $2a^2\operatorname{tr}(APA) < c$. Writing $2\operatorname{tr}(AT^2A) = 2a^2\operatorname{tr}(APA) + 2\operatorname{tr}(AQA)$ we have

$$m_j\{x : \|(aP + Q)x\| \leqq b\} > 1 - b. \qquad \text{Q.E.D.}$$

REFERENCES

1. L. Gross, *Integration and nonlinear transformations in Hilbert space*, Trans. Amer. Math. Soc., **94** (1960), 404–440.

2. L. Gross, *Harmonic analysis in Hilbert space*, Memoirs, American Mathematical Society, 46, 1963.

3. A. N. Kolmogorov, "A note to the papers of R. A. Minlos and V. Sazonov," *Teoriya Veroyatnostei i eyo Primenyeniya* **4** (1959), 237–239; *Theory of Probability and its Applications* **4** 221–223.

4. R. A. Minlos, *Generalized random processes and their extension to measures*, Trudy Moskov. Mat. Obsc., **8** (1959), 497–518.

5. Yu. V. Prohorov, *Convergence of random processes and limit theorems in probability theory, Theory of probability and its applications*, **1** (1956) English translation published by S. I. A. M., 157–214.

6. V. Sazonov, "A remark on characteristic Functionals," *Toriya Veroyatnostei i eyo Primenyeniya* **3** (1958), 201–205; *Theory of probability and its Applications* **3** 188–192.

7. I. E. Segal, *Tensor algebras over Hilbert space*, Trans. Amer. Math. Soc., **81** (1956), 106–134.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIFORNIA